

HOMOLOGICAL SUBSETS OF Spec

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ABSTRACT. We investigate homological subsets of the prime spectrum of a ring, defined by the help of the Ext-family $\{\text{Ext}_R^i(-, R)\}$. We extend Grothendieck's calculation of $\dim(\text{Ext}_R^g(M, R))$. We compute support of $\text{Ext}_R^i(M, R)$ in many cases. Also, we answer a low-dimensional case of a problem posed by Vasconcelos on the finiteness of associated prime ideals of $\{\text{Ext}_R^i(M, R)\}$. An application is given.

1. INTRODUCTION

Throughout the paper R is a commutative noetherian local ring of dimension d , and M a finitely generated module of grade g and dimension r , otherwise specializes. This paper deals with the invariants attached to the Ext-modules. Associated prime ideals of $\text{Hom}_R(-, \sim)$ computed several years ago. As far as we know, the first computation of $\text{Ass}(\underline{\text{Ext}}_{\mathcal{O}_X}^{\mathbf{g}_0}(\mathcal{F}, \mathcal{G}))$ appeared in the (LC) by Grothendieck. Here, (LC) referred to *local cohomology* [12] (also, see [13]). Set $\text{E-Ass}_R(M) := \bigcup_{i=0}^{\infty} \text{Ass}(\text{Ext}_R^i(M, R))$.

Problem 1.1. (Vasconcelos) Is $\text{E-Ass}_R(M)$ finite?

Denote the *homological support* by $\bigcup_i \text{Supp}(\text{Ext}_R^i(-, R))$. In Section 2 we show that the homological support have strange properties compared to the classical support. Despite of this, we show over finitely generated modules, homological support is the classical support. This drops $\text{p.dim}(M) < \infty$ from a result of Peskine and Szpiro, see [17]. As an application, we extend an implicit result of Grothendieck [12, 6.4.4]) by avoiding scheme-theory and (LC) (see Corollary 2.9):

Observation 1.2. If $\dim(\text{Ext}_R^{d-i}(M, R)) \leq i$ for all i and $g = d - r$, then $\dim(\text{Ext}_R^{d-r}(M, R)) = r$.

In the light of (LC) and in Proposition 2.12 we observe:

Observation 1.3. The formula $\dim(\text{Ext}_R^g(M, R)) = \dim M$ holds for all M if and only if R is Cohen-Macaulay.

In general, M is not supported in the support of $\text{Ext}_R^g(M, R)$, even over regular rings, see Example 2.10 i). However, we give situations for which M is supported in $\text{Supp}(\text{Ext}_R^g(M, R))$, see Corollary 2.7, 2.8, and Example 2.10. If we focus on modules of finite projective dimension over formally equidimensional rings, the game is changed: the dimension formula holds for all of such modules, see also [7, Proposition 2.2] by Beder.

Suppose R is a homomorphic image of a Gorenstein ring S . Recall from [23] that the *homological associated prime ideals* of M are defined by the set $\text{h-Ass}_R(M) := \bigcup_i \text{Ass}_R(\text{Ext}_S^i(M, S))$. Trivially, this is a finite set and coincides with the former $\text{E-Ass}_R(M)$ in the Gorenstein case. Following (LC), we observe in Section 3 that $\text{Ass}(M) \subseteq \text{h-Ass}(M)$. Set $M_{(i_1, \dots, i_p)p} := \text{Ext}_S^{i_1}(\text{Ext}_S^{i_2}(\dots(\text{Ext}_S^{i_p}(M, S), \dots, S), S)$.

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* All the results that we need from [12] are also in [13]. Concerning this we only cite to the english lecture note.

This comes from Vasconcelos investigation of the notions of *homological degree* and *well-hidden* associated prime ideals. We continue Section 3 to understand $\bigcup_{(i_1, \dots, i_p)} \text{Ass}(M_{(i_1, \dots, i_p)p})$. We determine it in the diagonal case with $p > 2$: $\bigcup_{p=0}^3 \text{Ass}(M_{(i, \dots, i)p}) = \bigcup_{p=0}^{\infty} \text{Ass}(M_{(i, \dots, i)p})$, see Corollary 3.12. Also, we determine it in the Cohen-Macaulay case (see Corollary 3.10).

Section 4 investigates different notions of *homological annihilators*. We connect them to the classical annihilator. The first one is the Bridger's ideal $\gamma(-) := \bigcap_{i \geq 0} \text{rad}(\text{Ann}_R \text{Ext}_R^i(-, R))$. The second one is the Auslander-Buchsbaum invariant factor $\alpha(-)$ [3]. The third one is $\text{h-Ann}(-) := \prod_{i=0}^d \text{Ann}_R \text{Ext}_R^i(-, R)$. By using homological support, we remark that $\gamma(-)$ is support sensitive in the category of finitely generated modules of positive grade. We derive a similar result for $\alpha(-)$. Motivated from Auslander's comments on the functor Ext , we collect several remarks on the annihilators of Ext -modules.

Section 5 deals with Problem 1.1. We first reduce it to the class of cyclic modules. We show a little more, please see Lemma 5.5. Then, we show

Observation 1.4. i) Problem 1.1 is true over 3-dimensional excellent normal local domains.

ii) Problem 1.1 is true over two dimensional reduced excellent local rings.

The final section motivated from a result of Macaulay at 1904. First, we recall a more general version of this by Serre, please see Theorem 6.1 and 6.2. These results presented in the Ext -form by Griffith and Evans: Let R be a regular local ring containing a field and I be a height two prime ideal such that $\text{Ext}_R^2(R/I, R)$ is cyclic. Then I is two generated.

Observation 1.5. Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring and I be a Cohen-Macaulay ideal of height two and of finite projective dimension. Then $\mu(\text{Ext}_R^2(R/I, R)) = \mu(I) - 1$.

2. HOMOLOGICAL INTERPRETATION OF SUPPORT

By $\text{p.dim}_R(-)$ (resp. $\text{id}_R(-)$) we mean projective (resp. injective) dimension. The notation $\text{E-Supp}(-)$ stands for $\bigcup_{i \geq 0} \text{Supp}(\text{Ext}_R^i(-, R))$. This may be empty for modules with quite large support:

Example 2.1. Let R be a complete local integral domain of positive dimension. Let F be the fraction field of R . It is shown by Auslander [1, Page 166] that $\text{Ext}_R^i(F, R) = 0$ for all i . So, $\text{E-Supp}(F) = \emptyset \neq \text{Supp}(F) = \text{Spec}(R)$.

Example 2.2. The complete-local assumption is important. Note that $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, \mathbb{Z})$ is related to the *adèle* groups from number theory. By accepting continuum hypothesis, one has $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, \mathbb{Z}) = \mathbb{R}$ as a vector space over \mathbb{Q} . So, $\text{E-Supp}_{\mathbb{Z}}(\mathbb{Q}) = \text{Supp}_{\mathbb{Z}}(\mathbb{Q})$.

Interestingly, may be homological support is quite large against to the classical support:

Example 2.3. Adopt the notation of Example 2.1. It is shown in [1, Page 166] that $R \simeq \text{Ext}_R^1(F/R, R)$. So,

$$\text{E-Supp}(F/R) = \text{Spec}(R) \supsetneq \text{Spec}(R) \setminus \{(0)\} = \text{Supp}(F/R).$$

Lemma 2.4. Let L and N be finitely generated and nonzero. Then $\text{Ext}_R^i(L, N) \neq 0$ for some $i \leq \dim N$.

Proof. For each ideal I recall that $\text{ht}_N(I)$ defines by $\inf\{\dim(N_{\mathfrak{p}}) : \mathfrak{p} \in \text{Supp}(N) \cap V(I)\}$, where $V(I)$ is the set of all prime ideals containing I . Since R is local, we have $L \otimes N \neq 0$. Consequently, $\text{Supp}(L) \cap \text{Supp}(N) \neq \emptyset$. Respell this as $\text{Supp}(N) \cap V(\text{Ann } L) \neq \emptyset$. Deduce from this that $\text{ht}_N(\text{Ann } L) < \infty$. It sufficient to recall that

$$\inf\{i : \text{Ext}_R^i(L, N) \neq 0\} = \text{grade}(\text{Ann } L, N) \leq \text{ht}_N(\text{Ann } L) < \infty.$$

□

Corollary 2.5. *Keep the above notation in mind. Then $\text{Supp}(L \otimes N) \subset \bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}_R^i(L, N))$.*

Proof. Let $\mathfrak{p} \in \text{Supp}(L \otimes N)$. Then $L_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ are nonzero. In view of Lemma 2.4, there is an $i \leq \dim N_{\mathfrak{p}} \leq \dim N$ such that $\text{Ext}_{R_{\mathfrak{p}}}^i(L_{\mathfrak{p}}, N_{\mathfrak{p}}) \neq 0$. Note that $\text{Ext}_R^i(L, N)_{\mathfrak{p}} \simeq \text{Ext}_{R_{\mathfrak{p}}}^i(L_{\mathfrak{p}}, N_{\mathfrak{p}})$, because of the finiteness of L and N . So, $\mathfrak{p} \in \bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}_R^i(L, N))$. □

Proposition 2.6. *Let M and N be finitely generated and nonzero. Then*

$$\text{Supp}(M \otimes N) = \bigcup_{i=0}^{\infty} \text{Supp}(\text{Ext}_R^i(M, N)) = \bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}_R^i(M, N)).$$

Proof. We bring the following trivial facts

- 1) $\text{Supp}(M \otimes N) = \text{Supp}(M) \cap \text{Supp}(N)$,
- 2) $\text{Supp}(\text{Ext}_R^i(M, N)) \subset \text{Supp}(M) \cap \text{Supp}(N)$, and
- 3) $\bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}_R^i(M, N)) \subset \bigcup_i \text{Supp}(\text{Ext}_R^i(M, N))$.

We look at

$$\begin{aligned} \text{Supp}(M \otimes N) &\stackrel{1}{=} \text{Supp}(M) \cap \text{Supp}(N) \\ &\stackrel{2}{\supseteq} \bigcup_i \text{Supp}(\text{Ext}_R^i(M, N)) \\ &\stackrel{3}{\supseteq} \bigcup_{i=0}^{\dim N} \text{Supp}(\text{Ext}_R^i(M, N)) \\ &\stackrel{2.5}{\supseteq} \text{Supp}(M \otimes N). \end{aligned}$$

The proof is now complete. □

M is called *quasi-perfect* if $\inf\{i : \text{Ext}_R^i(M, R) \neq 0\} = \sup\{i : \text{Ext}_R^i(M, R) \neq 0\}$.

Corollary 2.7. *Let M be quasi-perfect of grade g . Then $\text{Supp}(M) = \text{Supp}(\text{Ext}_R^g(M, R))$.*

Corollary 2.8. *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring, Gorenstein on its punctured spectrum, and M be an one-dimensional R -module. Then $\text{Ext}_R^{d-1}(M, R)$ has same support as of M . In particular, $\dim(\text{Ext}_R^{d-1}(M, R)) = 1$.*

Proof. We have $\text{id}(R_{\mathfrak{p}}) = \text{ht}(\mathfrak{p})$ for all non-maximal prime ideal \mathfrak{p} . This yields that $\text{Supp}(\text{Ext}_R^d(M, R)) \subset \{\mathfrak{m}\}$. We are going to apply the grade conjecture. This is the place that we use the Cohen-Macaulay assumption, because the conjecture verified over such a ring without any stress on the finiteness of projective dimension, see [17]. Thus,

$$\text{grade}(M) = \dim R - \dim M = d - 1,$$

e.g. $\text{Ext}_R^{d-1}(M, R) \neq 0$. In particular, \mathfrak{m} belongs to its support. In view of Proposition 2.6, we have

$$\begin{aligned} \text{Supp}(M) &= \text{Supp}(\text{Ext}_R^{d-1}(M, R)) \cup \text{Supp}(\text{Ext}_R^d(M, R)) \\ &= \text{Supp}(\text{Ext}_R^{d-1}(M, R)) \cup \{\mathfrak{m}\} \\ &= \text{Supp}(\text{Ext}_R^{d-1}(M, R)). \end{aligned}$$

This is what we want to prove. □

Corollary 2.9. *Let (R, \mathfrak{m}) be a d -dimensional local ring and M be r -dimensional. If $\dim(\text{Ext}_R^{d-i}(M, R)) \leq i$ for all i and that $g = d - r$, then $\dim(\text{Ext}_R^g(M, R)) = r$.*

The first condition holds when $\text{p. dim}(M) < \infty$ or R is Gorenstein. The second condition holds either R is Cohen-Macaulay or $\text{p. dim}(M) < \infty$ and R is formally equidimensional.

Proof. Set $\mathfrak{b}_i := \text{Ann}(\text{Ext}_R^i(M, R))$. In view of Proposition 2.6, $\text{Supp}(M) = V(\mathfrak{b}_g) \cup \dots \cup V(\mathfrak{b}_d)$. Let $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_r = \mathfrak{m}$ be a maximal and strict chain of prime ideals in the $\text{Supp}(M)$. We claim that $\mathfrak{p}_0 \in V(\mathfrak{b}_g)$. Suppose on the contradiction that $\mathfrak{p}_0 \notin V(\mathfrak{b}_g)$. Hence $\mathfrak{p}_0 \in V(\mathfrak{b}_\ell)$ for some $\ell > g$, i.e., $\mathfrak{p}_0 \supset \mathfrak{b}_\ell$. Clearly, $\mathfrak{p}_j \in V(\mathfrak{b}_\ell)$ for all j . By definition of \dim , we have $\dim(R/\mathfrak{b}_\ell) \geq r = d - g$. Due to our assumption, $V(\mathfrak{b}_\ell)$ is of dimension at most $d - \ell$. Combining these, we observe $d - g > d - \ell \geq \dim(R/\mathfrak{b}_\ell) \geq d - g$, which is a contradiction. Thus, $\mathfrak{p}_0 \in V(\mathfrak{b}_g)$. Therefore, $r = \dim(M) \geq \dim(\text{Ext}_R^g(M, R)) \geq r$. So, $\dim(\text{Ext}_R^g(M, R)) = r$. \square

Example 2.10. i) Let $R := \mathbb{Q}[[x, y, z]]$ and let $I := (xy, xz)$. Then $\text{Supp}(R/I) \neq \text{Supp}(\text{Ext}_R^g(R/I, R))$.

ii) Let R be Gorenstein and $L := \text{Ext}_R^i(\text{Ext}_R^i(M, R), R)$ for each i . Then $\text{Supp}(\text{Ext}_R^i(L, R)) = \text{Supp}(L)$.

iii) Let R be Gorenstein and M a Cohen-Macaulay module of grade g . Then $\text{Supp}(\text{Ext}_R^g(M, R)) = \text{Supp}(M)$.

Proof. i) Set $M := R/I$. We look at the minimal free resolution of M :

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} -z & y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} xy & xz \end{bmatrix}^t} R.$$

Apply $\text{Hom}_R(-, R)$ to it, we get to the following complex

$$R \xrightarrow{\begin{bmatrix} xy & xz \end{bmatrix}^t} R^2 \xrightarrow{\begin{bmatrix} -z & y \end{bmatrix}} R \longrightarrow 0.$$

Note that $\text{Ass}(\text{Hom}_R(M, R)) = \text{Supp}(M) \cap \text{Ass}(R) = \emptyset$. Let us compute the $\text{Ext}_R^1(M, R)$. As y, z is a regular sequence in R , the Koszul complex on $\{y, z\}$ presents the free resolution of (y, z) . Thus, the kernel of $(-z, y) : R^2 \rightarrow R$ is generated by (y, z) and so $\text{Ext}_R^1(M, R) \simeq (y, z)R/(xy, xz)R$. This yields that $\text{Ann}(\text{Ext}_R^1(M, R)) = (x)$. Therefore, $g := \text{grade}(M) = 1$ and $(y, z) \in \text{Supp}(M) \setminus V(x)$. We deduce from this that $\text{Supp}(\text{Ext}_R^g(M, R)) = V(x) \subsetneq \text{Supp}(M)$.

ii) This follows by [4, 7.60], where Bridger has shown the following amusing result:

$$L = \text{Ext}_R^i(\text{Ext}_R^i(M, R), R) \simeq \text{Ext}_R^i(\text{Ext}_R^i(\text{Ext}_R^i(\text{Ext}_R^i(M, R), R), R), R) = \text{Ext}_R^i(\text{Ext}_R^i(L, R), R).$$

iii) This follows from the Ext-duality [6, 3.3.10]: $M \simeq \text{Ext}_R^g(\text{Ext}_R^g(M, R), R)$. \square

Discussion 2.11. (Grothendieck [12, 6.4.4]) Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with a canonical module and M a finitely generated module. Then $\dim(\text{Ext}_R^{d-\dim M}(M, \omega_R)) = \dim M$.

Proposition 2.12. *Let (R, \mathfrak{m}) be a local ring and M a finitely generated module of grade g . Then $\dim(\text{Ext}^g(M, R)) = \dim M$ for all M if and only if R is Cohen-Macaulay.*

Proof. Suppose first that R is not Cohen-Macaulay. We show the dimension formula does not hold for the residue field. Note that $g := \text{grade}(R/\mathfrak{m}) = \text{depth}(R)$. So $\dim(\text{Ext}_R^g(R/\mathfrak{m}, R)) = 0 \not\geq d - g$.

Suppose now that R is Cohen-Macaulay. In order to prove the formula without loss of generality, we assume that R is complete. Then R has a canonical module [6, Theorem 3.3.6]. By Discussion 2.11, $\dim(\text{Ext}_R^{d-\dim M}(M, \omega_R)) = \dim M$. In view of the grade conjecture over Cohen-Macaulay rings and without any stress about finiteness of projective dimension we have $g = d - \dim M$. Let $\underline{x} := x_1, \dots, x_g$ be a maximal R -sequence in $\text{Ann}(M)$. Put $\overline{R} := R/\underline{x}R$ and note that M can view as an \overline{R} -module. As

ω_R is maximal Cohen-Macaulay, \underline{x} is a ω_R -sequence in $\text{Ann}(M)$. Due to the Rees lemma [16, Page 140], there is the isomorphism

$$\text{Ext}_R^g(M, \omega_R) \simeq \text{Ext}_R^0(M, \omega_R / \underline{x}\omega_R) \simeq \text{Ext}_R^0(M, \omega_{\overline{R}}).$$

The last isomorphism is in [6, Theorem 3.3.5], e.g., $\omega_{\overline{R}} \simeq \omega_R / \underline{x}\omega_R$. In a similar vein there is an isomorphism $\text{Ext}_R^g(M, R) \simeq \text{Ext}_{\overline{R}}^0(M, \overline{R})$.

Claim: One has $\dim(\text{Hom}_{\overline{R}}(M, \overline{R})) = \dim(\text{Hom}_{\overline{R}}(M, \omega_{\overline{R}}))$. To this end we show they have the same associated prime ideals. As \overline{R} and $\omega_{\overline{R}}$ are Cohen-Macaulay over \overline{R} , their associated prime ideals are the minimal primes of their support. On the other hand \overline{R} and $\omega_{\overline{R}}$ have a same set as the support, because $(\omega_{\overline{R}})_{\mathfrak{p}} = \omega_{\overline{R}_{\mathfrak{p}}}$, see [6, Theorem 3.3.5]. Thus, $\text{Ass}_{\overline{R}}(\overline{R}) = \text{Ass}_{\overline{R}}(\omega_{\overline{R}})$. In view of [6, Ex. 1.2.27],

$$\text{Ass}_{\overline{R}}(\text{Hom}_{\overline{R}}(M, \overline{R})) = \text{Supp}_{\overline{R}}(M) \cap \text{Ass}_{\overline{R}}(\overline{R}) = \text{Supp}_{\overline{R}}(M) \cap \text{Ass}_{\overline{R}}(\omega_{\overline{R}}) = \text{Ass}_{\overline{R}}(\text{Hom}_{\overline{R}}(M, \omega_{\overline{R}})).$$

Combining these we get

$$\begin{aligned} \dim M &= \dim(\text{Ext}_R^g(M, \omega_R)) \\ &= \dim(\text{Hom}_{\overline{R}}(M, \omega_{\overline{R}})) \\ &= \dim(\text{Hom}_{\overline{R}}(M, \overline{R})) \\ &= \dim(\text{Ext}_R^g(M, R)). \end{aligned}$$

□

3. HOMOLOGICAL ASSOCIATED PRIMES

Example 3.1. Let R be either $\mathbb{Q}[x, y, z]$ or $\mathbb{Q}[[x, y, z]]$ and let $M := R/(xy, xz)$. Then

$$\bigcup \text{Min}(\text{Ext}_R^i(M, R)) = \text{Ass}_R(M) = \{(x), (y, z)\}.$$

Proof. In view of Example 2.10, we have:

$$\text{Ext}_R^i(M, R) \simeq \begin{cases} \frac{(-y, z)}{(xy, xz)} & \text{if } i = 1 \\ \frac{R}{(y, z)} & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Thus, } \bigcup \text{Min}(\text{Ext}_R^i(M, R)) = \text{Ass}(M).$$

□

Discussion 3.2. (Grothendieck) Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension d with a canonical module. Let M be a finitely generated module of dimension n and depth t . The following holds.

- i) $\text{Ext}_R^i(M, \omega_R) = 0$ for $i \notin [d - n, d - t]$. Also, $\text{Ext}_R^i(M, \omega_R) \neq 0$ for $i = d - n$ and $i = d - t$.
- ii) $\dim(\text{Ext}_R^i(M, \omega_R)) \leq d - i$.

The proof of Discussion 3.2(ii) implies the following more general fact:

Proposition 3.3. *Let M and N be finitely generated modules such that either $\text{p. dim}(M) < \infty$ or $\text{id}(N) < \infty$ over any commutative noetherian ring. Let $i \leq d$. Then $\dim(\text{Ext}_R^i(M, N)) \leq d - i$.*

Proof. Without loss of generality we assume that $d := \dim R$ is finite. Suppose \mathfrak{p} is of coheight $> d - i$. Suppose first that $\text{id}(N) < \infty$. Due to a theorem of Bass [6, Theorem 3.1.17], we observe that

$$\text{id}(N_{\mathfrak{p}}) \leq \text{depth}(R_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} = \text{ht}(\mathfrak{p}) \leq d - \dim R/\mathfrak{p} < d - (d - i) = i.$$

Thus $\text{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$. Suppose now that $\text{p. dim}(M) < \infty$. By Auslander-Buchsbaum formula and in a similar way as above, $\text{p. dim}(M_{\mathfrak{p}}) < i$. Consequently, $\text{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$. We show in each cases that $\text{Supp}(\text{Ext}_R^i(M, N))$ is of dimension less or equal than $d - i$. This proves the desired fact. □

Example 3.4. i) The finitely generated assumption is really needed. In view of Example 2.2, $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, \mathbb{Z})$ is a nonzero rational vector space. So, $\dim(\text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, \mathbb{Z})) = 1 > 0 = d - 1$.

ii) The finiteness of homological dimensions is important. Look at $R := \mathbb{Q}[[X, Y]]/(X^2)$. It is easy to see $\text{Ext}_R^i(R/xR, R/xR) \simeq R/xR$ for all i . So, $\dim(\text{Ext}_R^1(R/xR, R/xR)) = 1 > 0 = d - 1$.

In the above example $\text{Ext}_R^i(R/(x), R) = 0$ for all $i > 0$. One may search the validity of $\dim(\text{Ext}_R^i(-, R)) \leq d - i$. In general, this is not the case as the next example says.

Example 3.5. Let $R := \mathbb{Q}[X, Y, Z]/(X^2, XY, XZ)$. This is a 2-dimensional ring and $\min(R) = \{(x)\}$. Due to the formula $\text{rad}(y, z) = (x, y, z)$, one can show that $\{y, z\}$ is a system of parameters. Since $\{y, z\}$ is not a regular sequence, R is not Cohen-Macaulay. Set $M := R/xR$. Note that neither $\text{p. dim}(M) < \infty$ nor $\text{id}(R) < \infty$. We are going to show $\dim(\text{Ext}_R^2(M, R)) = 2 > 0 = d - 2$. We restate the 3 relations $x^2 = xy = xz = 0$ of $\{x, y, z\}$ by

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We restate the Koszul relations of $\{x, y, z\}$ by

$$\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we put all of these relations of $\{x, y, z\}$ to the following package

$$A := \begin{pmatrix} x & 0 & 0 & -y & -z & 0 \\ 0 & x & 0 & x & 0 & -z \\ 0 & 0 & x & 0 & x & y \end{pmatrix}.$$

Look at the free resolution of M :

$$\cdots \longrightarrow R^6 \xrightarrow{A} R^3 \xrightarrow{\begin{bmatrix} x & y & z \end{bmatrix}} R \xrightarrow{x} R \longrightarrow M \longrightarrow 0.$$

Delete M from the right and apply $\text{Hom}_R(-, R)$ we get to the following complex

$$0 \longrightarrow R \xrightarrow{x} R \xrightarrow{\begin{bmatrix} x & y & z \end{bmatrix}^t} R^3 \xrightarrow{A^t} R^6.$$

Let $\{a, b, c\}$ be such that $[a, b, c]A = 0$. It is solution of the following system of six equations

$$\begin{cases} ax = bx = cx = 0 \\ -ay + bx = 0 \\ -az + cx = 0 \\ -bz + cy = 0 \end{cases}$$

It is easy to see that $\{(x, 0, 0), (0, x, 0), (0, 0, x), (0, y, z)\}$ are the solutions. Hence

$$\text{Ext}_R^2(M, R) = \frac{\ker(A^t)}{\text{im}(\begin{bmatrix} x & y & z \end{bmatrix}^t)} \supseteq \frac{\langle (x, 0, 0), (0, x, 0), (0, 0, x), (0, y, z) \rangle}{(x, y, z)R}.$$

Thus,

$$(x) \subseteq \text{Ann}(\text{Ext}_R^2(R/(x), R)) \subseteq \text{Ann}\left(\frac{\langle (x, 0, 0), (0, x, 0), (0, 0, x), (0, y, z) \rangle}{(x, y, z)R}\right) = (x).$$

We observe that $\dim(\text{Ext}_R^2(M, R)) = \dim R/(x) = 2$.

Example 3.1 extends in the following sense.

Proposition 3.6. *Let (R, \mathfrak{m}) be a local ring which is a homomorphic image of a Gorenstein ring (S, \mathfrak{n}) of dimension d . Let M be a finitely generated R -module of dimension n . Then $\text{Ass}_R(M) \subseteq \bigcup_{i=d-n}^n \text{Min}_R(\text{Ext}_S^i(M, S))$. The equality holds if M is Cohen-Macaulay.*

Proof. We use an idea taking from Grothendieck [12, Proposition 6.6]. By $(-)^v$ we mean the Matlis dual functor. We view M as an S -module via the map $S \rightarrow R$. In the light of the local duality theorem, we have $H_{\mathfrak{m}}^i(M)^v \simeq \text{Ext}_S^{d-i}(M, S)$. Also, $H_{\mathfrak{m}}^i(M) \simeq H_{\mathfrak{n}}^i(M)$ as an R -module. Since $H_{\mathfrak{m}}^i(M)^{vv} \simeq H_{\mathfrak{m}}^i(M)$ we observe that

$$\mathfrak{a}_i(M) := \text{Ann}_R(H_{\mathfrak{m}}^i(M)) = \text{Ann}_R(H_{\mathfrak{n}}^i(M)) = \text{Ann}_R(\text{Ext}_S^{d-i}(M, S)).$$

In view of Discussion 3.2(ii), $\dim \text{Ext}_S^{d-i}(M, S) \leq i$. Let $\mathfrak{p} \in \text{Ass}_R(M)$ of dimension i and let \mathfrak{q} be its image in S . Then $\dim(S_{\mathfrak{q}}) = d - i$. By Discussion 3.2(i)

$$\begin{aligned} \mathfrak{p} \in \text{Ass}(M) &\iff \text{depth}(M_{\mathfrak{p}}) = 0 \\ &\iff \text{depth}(M_{\mathfrak{q}}) = 0 \\ &\stackrel{(+)}{\implies} \text{Ext}_{S_{\mathfrak{q}}}^{d-i}(M_{\mathfrak{q}}, S_{\mathfrak{q}}) \neq 0 \\ &\iff \text{Ext}_S^{d-i}(M, S)_{\mathfrak{p}} \neq 0 \\ &\iff \mathfrak{p} \in V(\mathfrak{a}_i(M)). \end{aligned}$$

We remark that $(+)$ is "if and only if" when M is Cohen-Macaulay. As minimal elements of support are the associated primes, we get the claim by recalling again that $\dim(\text{Ext}_S^{d-i}(M, S)) \leq i$. \square

Example 3.7. i) The finitely generated assumption is important: Let R be a complete regular local ring of positive dimension with the fraction field F . Clearly, $\text{Ass}(F) \neq \emptyset$. Therefore, $\text{Ass}(F) \not\subseteq \text{h-Ass}(F) = \emptyset$.

ii) The Cohen-Macaulayness is important: We revisit Example 3.1. We take $R = S$. Note that $\emptyset \neq \text{Ass}(I) \subset \text{Ass}(R) = \{0\}$, i.e., $\text{Ass}(I) = \{0\}$. In view of Example 2.10, we observe that

$$(y, z) \in \left(\bigcup_{i=d-n}^n \text{Min}_R(\text{Ext}_S^i(I, S)) \right) \setminus \text{Ass}_R(I).$$

Corollary 3.8. *Let M be finitely generated. Then $\text{Ass}(M) \subseteq \text{h-Ass}(M)$.*

Recall from [22] the *hidden associated prime ideals* of M is $\text{Hidd}_R(M) := \text{h-Ass}_R(M) \setminus \text{Ass}_R(M)$. Now, we recover a result of Foxby [10, Proposition 3.4b] via two different arguments:

Corollary 3.9. *Let R be a Gorenstein local ring and M a Cohen-Macaulay module. Then $\text{Hidd}(M) = \emptyset$.*

First proof. As M is Cohen-Macaulay, $\text{Ext}_R^i(M, R)$ is either zero or Cohen-Macaulay. This may be respell by $\text{Min}(\text{Ext}_R^i(M, R)) = \text{Ass}_R(\text{Ext}_R^i(M, R))$. Proposition 3.6 yields the claim.

Second proof. Apply Corollary 2.9.

We drop the later p from $(i_1, \dots, i_p)p$ if there is no danger of confusion.

Corollary 3.10. *Let (R, \mathfrak{m}) be a Gorenstein and M Cohen-Macaulay. Then $\bigcup_p \bigcup_{(i_1, \dots, i_p)} \text{Ass}(M_{(i_1, \dots, i_p)}) = \text{Ass}(M)$.*

Example 3.11. We revisit Example 2.10. Let $S := \mathbb{Q}[[x, y, z]]$ and let $M := S/(xy, xz)$. Let $p \geq 1$. Then

$$M_{(i_1, \dots, i_{p-1}, 2)} := \begin{cases} S/(y, z) & \text{if } i_1 = \dots = i_{p-1} = 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. In view of Example 2.10, $\text{Ext}_S^2(M, S) \simeq \frac{S}{(y,z)}$. Since y, z is a regular sequence on S , from its Koszul complex we obtain that $\text{Ext}_S^i(\text{Ext}_S^2(M, S), S) \simeq \frac{S}{(y,z)} \delta_{2,i}$, where δ is the Kronecker delta. From this we get the claim. \square

Corollary 3.12. *Let R be a Gorenstein local ring, M a finitely generated module and i be any integer. Then for any $n > 2$ we have $\bigcup_{p=0}^3 \text{Ass}(M_{(i,\dots,i)p}) = \bigcup_{p=0}^n \text{Ass}(M_{(i,\dots,i)p})$.*

Proof. We revisit Bridger's amusing formula $M_{(i,i)} \simeq M_{(i,i,i,i)}$. Apply $\text{Ext}_R^i(-, R)$ to this, $M_{(i,\dots,i)5} \simeq M_{(i,\dots,i)3}$. Similarly, we detect $M_{(i,\dots,i)p}$ from $M_{(i,\dots,i)q}$ for some $q < p$. The induction completes the proof. \square

4. HOMOLOGICAL ANNIHILATORS

In the Introduction we attached three types of ideals to a module, namely $\gamma(-)$, $\text{h-Ann}(-)$ and $\alpha(-)$. Let us present the definition of $\alpha(M) := \bigcap_{\text{Ann}(\wedge^i(M)) \neq 0} \text{Ann}(\wedge^i(M))$. Also, $\beta(M) := \{x \in R : M_x \text{ is } R_x\text{-projective}\}$ introduced by Auslander and Buchsbaum. First, we compute things via an example.

Example 4.1. Let $R := \mathbb{Q}[[x, y, z]]$ and set $M := R/(xy, xz)$. One has

$$\gamma(M) = \text{Ann}(M) = \text{h-Ann}(M) = \beta(M) = \alpha(M).$$

Also, $\gamma(xy, xz) \neq \text{Ann}(xy, xz)$.

Proof. In the case of modules of finite projective dimension, the claims $\gamma(M) = \beta(M) = \alpha(M)$ is in [5]. In view of Example 2.10, $\gamma(M) = \text{Ann}(M) = \text{h-Ann}(M)$. Again, Example 2.10 implies that $\gamma(xy, xz) \neq \text{Ann}(xy, xz)$. \square

Proposition 4.2. *Let M be a finitely generated module and of positive grade. Then $\gamma(M) = \text{rad}(\text{Ann}(M))$.*

Proof. The grade condition says that $\text{Ext}_R^0(M, R) = 0(+)$. Set $\mathfrak{c}_i(M) = \text{rad}(\text{Ann}(\text{Ext}_R^i(M, R)))$. In view of Proposition 2.6 we conclude that

$$\begin{aligned} V(\text{Ann}(M)) &= \text{Supp}(M) \\ &\stackrel{(+)}{=} \bigcup_{i=1}^d \text{Supp}(\text{Ext}_R^i(M, R)) \\ &= \bigcup_{i=1}^d V(\mathfrak{c}_i(M)) \\ &= V(\bigcap_{i=1}^d \mathfrak{c}_i(M)). \end{aligned}$$

Since $\text{Ann}(M) \subset \bigcap_{i=1}^d \mathfrak{c}_i(M)$, we get from the displayed item that $\bigcap_{i=1}^d \mathfrak{c}_i(M) = \text{rad}(\text{Ann}(M))$. Again we look at $\text{rad}(\text{Ann}(M)) \subset \mathfrak{c}_i(M)$. Immediately, we deduce that

$$\text{rad}(\text{Ann}(M)) \subset \bigcap_{i=1}^{\infty} \mathfrak{c}_i(M) \subset \bigcap_{i=1}^d \mathfrak{c}_i(M) = \text{rad}(\text{Ann}(M)).$$

Consequently, $\gamma(M) = \bigcap_{i=1}^{\infty} \mathfrak{c}_i(M) = \text{rad}(\text{Ann}(M))$. \square

Example 4.3. The finitely generated assumption is important: Adopt the notation of Example 2.1. Note that $\text{Hom}_R(F, R) = 0$. Thus F is of positive grade and $\gamma(F) = R$. It remains to note that $\text{Ann}(F) = 0$.

Corollary 4.4. *Let M and N be modules of positive grade with the same support. Then $\gamma(M) = \gamma(N)$.*

Corollary 4.5. *Let M be a finitely generated module of finite projective dimension. Then $\gamma(M) = \text{rad}(\text{Ann}(M))$ if and only if M is of positive grade.*

Proof. One direction is in Proposition 4.2, without any assumption on the projective dimension. To see the other side implication, we may assume that $\gamma(M) \neq \text{rad}(\text{Ann}(M))$. Since $\text{rad}(\text{Ann}(M)) \subsetneq \gamma(M)$, there is an M -regular element in $\gamma(M)$. In view of [4, Theorem 7.57], M is a 1-syzygy module. It turns out that M is subset of a free module. So, $\text{Hom}_R(M, R) \neq 0$, i.e., $\text{grade}(M)$ is zero. \square

Corollary 4.6. *Let M and N be modules of positive grade and of finite projective dimension with the same support. Then $\beta(M) = \beta(N)$.*

Proof. In view of [4, Proposition 4], $\beta(M) = \gamma(M)$. So, the claim follows from the support sensitivity of $\gamma(-)$. \square

Discussion 4.7. i) By Proposition 2.6, $\text{h-Ann}(M)$ and $\text{Ann}(M)$ have a same radical.

ii) Over Gorenstein local rings $\text{h-Ann}(M) \subset \text{Ann}(M)$. To see this its enough to apply local duality along with [18, Page 350]. We should remark that, over *polynomial rings*, this is in [14, Page 215, Remark], where it referred to an unpublished work of Eisenbud and Evans.

iii) Suppose (R, \mathfrak{m}) is Gorenstein and local. Then $\text{Ann}(M)^{\text{Gdim}(M)-g+1} \subseteq \text{h-Ann}(M) \subseteq \text{Ann}(M)$.

Corollary 4.8. *Suppose R is Gorenstein and M is quasi-perfect. Then $\text{h-Ann}(M) = \text{Ann}(M)$.*

Example 4.9. The quasi-perfect assumption is important: Suppose (R, \mathfrak{m}) is a polynomial ring over an infinite field and of dimension $d > 3$. Let $2 \leq i_1 < \dots < i_\ell \leq d - 2$. Evans and Griffith [9, Theorem A] constructed a prime ideal \mathfrak{p} which is not maximal such that $\text{Ext}_R^{d-i+1}(R/\mathfrak{p}, R) \simeq R/\mathfrak{m}$ for $i \in \{i_1, \dots, i_\ell\}$ and zero exactly when $i \notin \{i_1, \dots, i_\ell, d - 1\}$. The undetermined Ext is $\text{Ext}_R^2(R/\mathfrak{p}, R)$. Put $\mathfrak{a}_2 := \text{Ann}(\text{Ext}_R^2(R/\mathfrak{p}, R))$. By Observation 1.2, $\dim(R/\mathfrak{a}_2) = d - 2$. As \mathfrak{p} is a $(d - 2)$ -dimensional prime ideal and $\mathfrak{p} \subseteq \mathfrak{a}_2$, we have $\mathfrak{a}_2 = \mathfrak{p}$. Thus, $\text{h-Ann}(R/\mathfrak{p}) = \mathfrak{m}^\ell \mathfrak{p} \neq \mathfrak{p} = \text{Ann}(R/\mathfrak{p})$.

Remark 4.10. If M is finitely generated over an integral domain, then $0 \neq \gamma(M)$. Indeed, by generic of freeness, there is $a \in R$ such that M_a is free as an R_a -module. Let x be a suitable power of a . Due to [5], $x \text{Ext}_R^i(M, R) = 0$ for all $i > 0$. In particular, $\gamma(M)$ and $\beta(M)$ are nonzero.

Example 4.11. With the same notation as of Example 2.1, $\text{Ext}_R^1(F/R, R) \simeq R$. Thus, $\gamma(F/R) = 0$.

Let J be the jacobian ideal of a complete local ring R . Wang gives an $s \in \mathbb{N}$ such that $J^s \text{Ext}_R^i(M, R\text{-mod}) = 0$ for all $i > \dim R$, where $R\text{-mod}$ is the category of finitely generated modules. It may be $J = 0$. For example, $R := \mathbb{Q}[X]/(X^2)$.

Lemma 4.12. *Let M be a finitely generated module over any commutative ring and let L be any module. Then $\text{Ext}_R^i(M, L)$ has no nonzero projective submodule.*

Proof. We use the induction. The case $i = 1$ is in [1, Theorem 4.1]. We make use of the standard shifting isomorphisms. Let $E(L)$ be the injective envelope of L . For the higher ext, its enough to look at $0 \rightarrow L \rightarrow E(L) \rightarrow E(L)/L \rightarrow 0$ and apply the induction hypothesis throughout natural identification $\text{Ext}_R^{i-1}(M, E(L)/L) \simeq \text{Ext}_R^i(M, L)$ for all $i > 1$. \square

Corollary 4.13. *Let M be finitely generated and L be any module. Then each element of $\text{Ext}_R^i(M, L)$ has a nontrivial annihilation for all $i > 0$.*

Proof. Suppose there is $x \in \text{Ext}_R^i(M, L)$ which is a faithful element. Then $R = \frac{R}{(0:x)} \simeq xR \hookrightarrow \text{Ext}_R^i(M, L)$. By Lemma 4.12, $\text{Ext}_R^i(M, L)$ has no nonzero projective submodule. This yields the contradiction. \square

Corollary 4.14. *Let R be reduced and M finitely generated. Then $\text{Ass}(R) \cap \text{E-Ass}(M) = \text{Ass}(R) \cap \text{Supp}(M)$.*

Proof. Suppose there is $\mathfrak{p} \in \text{Ass}(R) \cap \text{E-Ass}(M)$. Then $\mathfrak{p} = (0 : r)$ for some $r \in R$ and R/\mathfrak{p} embedded into $\text{Ext}_R^i(M, R) = \text{Ext}_R^1(\Omega^{i-1}(M), R)$. Here $\Omega^{i-1}(M)$ is the $(i-1)$ -th syzygy module of M . Suppose $i \neq 0$. By [1, Corollary 4.2], trace of R/\mathfrak{p} is nilpotent. As R is reduced, $\text{tr}(R/\mathfrak{p}) = 0$. But $0 \neq r \in \text{tr}(R/\mathfrak{p})$. This contradiction says that $i = 0$. Since $\text{Ass}(\text{Ext}_R^0(M, R)) = \text{Ass}(R) \cap \text{Supp}(M)$, we get the claim. \square

5. COMMENTS ON $\text{E-Ass}(-)$

Lemma 5.1. *Let R be any commutative ring and $I \triangleleft R$. Then $\text{Ext}_R^i(I, -) \simeq \text{Ext}_R^{i+1}(R/I, -)$ for all $i > 0$. In particular, $I \subseteq \gamma(I)$.*

Proof. Look at the induced long exact sequence of Ext-modules induced from the short exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$. \square

We start by 3 reductions of Problem 1.1. The first reduction is a weak inductive method.

Proposition 5.2. *Let R be a d -dimensional local ring such that Problem 1.1 is true over $d-1$ dimensional rings. Then $|\text{E-Ass}(I)| < \infty$ for all ideal I of positive grade.*

Proof. Recall from Lemma 5.1 that $I \text{Ext}_R^{>0}(I, R) = 0$. Let $y \in I$ be a regular element. Look at the exact sequence

$$0 \longrightarrow I \xrightarrow{y} I \longrightarrow I/yI \longrightarrow 0.$$

This induces the exact sequence

$$0 \longrightarrow \text{Ext}_R^i(I, R) \longrightarrow \text{Ext}_R^{i+1}(I/yI, R) \longrightarrow \text{Ext}_R^{i+1}(I, R) \longrightarrow 0.$$

Thus $|\text{E-Ass}_R(I)| < \infty$ provided $|\text{E-Ass}_R(I/yI)| < \infty$. By [16, Page 140], there is the isomorphism

$$\text{Ext}_R^{i+1}(I/yI, R) \simeq \text{Ext}_{\overline{R}}^i(I/yI, \overline{R}),$$

we observe that $\text{E-Ass}_R(I)$ is finite if $\text{E-Ass}_{\overline{R}}(I/yI)$ is finite. \square

Remark 5.3. i) Problem 1.1 reduces to complete case.

ii) Problem 1.1 reduces to maximal Cohen-Macaulay modules, when R is Cohen-Macaulay.

Proof. i) Let \mathfrak{p} be a prime ideal and let $\mathfrak{q} \in \text{Ass}(\widehat{R}/\mathfrak{p}\widehat{R})$. Indeed, if $x \in R \setminus \mathfrak{p}$, then $R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p}$ is injective and so $\widehat{R}/\mathfrak{p}\widehat{R} \xrightarrow{x} \widehat{R}/\mathfrak{p}\widehat{R}$ is injective. Thus, $x \notin \bigcup_{\mathfrak{Q} \in \text{Ass}(\widehat{R}/\mathfrak{p}\widehat{R})} \mathfrak{Q}$. Therefore, $\mathfrak{q} \cap R = \mathfrak{p}$. This show that if $\mathfrak{p}_1 \neq \mathfrak{p}_2$ then $\text{Ass}(\widehat{R}/\mathfrak{p}_1\widehat{R}) \neq \text{Ass}(\widehat{R}/\mathfrak{p}_2\widehat{R})$ (*). Now suppose that $\bigcup_i \text{Ass}_{\widehat{R}}(\text{Ext}_R^i(M \otimes \widehat{R}, \widehat{R}))$ is finite. In view of [16, Theorem 23.2],

$$\text{Ass}_{\widehat{R}}(\text{Ext}_R^i(M, R) \otimes_R \widehat{R}) = \bigcup_{\mathfrak{p} \in \text{Ass Ext}_R^i(M, R)} \text{Ass}(\widehat{R}/\mathfrak{p}\widehat{R}).$$

Combine this along with (*) we deduce that $\bigcup_i \text{Ass}(\text{Ext}_R^i(M, R))$ is finite.

ii) Assume that $\dim R = d$ and M be nonzero. We assume that $\text{p. dim}(M) = \infty$. We look at the following exact sequence:

$$0 \longrightarrow \Omega_d \longrightarrow F_{d-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_i is finitely generated free for all $i = 0, \dots, d$. Then $\Omega_d \neq 0$. In view of [6, Exercise 2.1.26] Ω_d is maximal Cohen-Macaulay. Also, $\text{Ext}_R^{i+d}(M, R) \simeq \text{Ext}_R^i(\Omega_d, R)$. It turns out that finiteness of $\text{E-Ass}_R(\Omega_n)$ implies the finiteness of $\text{E-Ass}_R(M)$. Without loss of the generality, we may assume that M is maximal Cohen-Macaulay. \square

We say \mathfrak{a} has *max-height* at most t if for all $\mathfrak{p} \in \text{Ass}(R/\mathfrak{a})$, one has $\text{ht}(\mathfrak{p}) \leq t$.

Theorem 5.4. ([2, Theorem B.b]) *Let R be an integrally closed Cohen-Macaulay ring and N a finitely generated R -module. There is an ideal \mathfrak{b} of max-height at most 2 such that $\text{Ext}_R^i(\mathfrak{b}, -) \simeq \text{Ext}_R^{i+2}(N, -) \forall i \geq 2$.*

Let us reduce things to the cyclic modules.

Lemma 5.5. *Let R be an integrally closed Cohen-Macaulay local ring. The following are equivalent:*

- i) $|\text{E-Ass}(M)| < \infty$ for all finitely generated M .
- ii) $|\text{E-Ass}(\mathfrak{b})| < \infty$ for all ideal \mathfrak{b} of max-height at most two.
- iii) $|\text{E-Ass}(R/\mathfrak{b})| < \infty$ for all ideal \mathfrak{b} of max-height at most two.

Proof. Lemma 5.1 says that $\text{Ext}_R^i(\mathfrak{b}, -) \simeq \text{Ext}_R^{i+1}(R/\mathfrak{b}, -)$ for all $i > 0$. By Auslander's remark and for all $i \geq 2$, we have $\text{Ext}_R^{i+1}(R/\mathfrak{b}, -) \simeq \text{Ext}_R^{i+2}(M, -)$. We note that the initial terms $\text{Ext}_R^i(M, -)$ are not effective, because the associated prime ideals of a finitely generated module is a finite set. This yields the claim. \square

Fact 5.6. i) The set $|\text{E-Ass}(M)| < \infty$, if $\text{p.dim}(M) < \infty$. The same claim is true if $\text{Gdim}(M) < \infty$.
 ii) The problem 1.1 is true if R is Gorenstein on its punctured spectrum.

Proof. i) The first claim is clear. Recall that if $\text{Gdim}(M) < \infty$, then $\text{Gdim}(M) = \sup\{i : \text{Ext}_R^i(M, R) \neq 0\}$. So, $\text{E-Ass}(M)$ is finite.

ii) Let $i \geq \dim R$, which is finite as we may assume that R is local. Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal. Since $R_{\mathfrak{p}}$ is Gorenstein, $\text{id}(R_{\mathfrak{p}}) = \text{ht}(\mathfrak{p}) < \infty$. Thus, $\text{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$. Consequently, $\text{Supp}(\text{Ext}_R^i(M, R)) \subset \{\mathfrak{m}\}$. Then $\text{E-Ass}(M) \subset \bigcup_{i=0}^{\dim R-1} \text{Ass}(\text{Ext}_R^i(M, R)) \cup \{\mathfrak{m}\}$. The later is a finite set. So, $\text{E-Ass}(M)$ is a finite set. \square

Corollary 5.7. *Problem 1.1 is true over two dimensional normal local domains.*

Proof. This follows by Fact 5.6 ii), because $R_{\mathfrak{p}}$ is regular for all prime ideal \mathfrak{p} of height one. \square

Proposition 5.8. i) *Problem 1.1 is true over 3-dimensional excellent normal local domains.*

ii) *Problem 1.1 is true over two dimensional reduced excellent local rings.*

Proof. Let us bring a general phenomena. The Gorenstein locus $\text{Gor}(X)$ of $X := \text{Spec}(R)$ is the set of all primes \mathfrak{p} such that $R_{\mathfrak{p}}$ is Gorenstein. Since R is excellent and in view of [11], there is an ideal I such that $X \setminus \text{Gor}(X) = V(I)$.

i) Normal rings are $R(1)$. As R is normal, $\text{ht}(I) > 1$. Due to $\dim R = 3$ we get that $\dim R/I \leq 1$. Thus, for all $i \geq 3$ we have $\text{Supp}(\text{Ext}_R^i(M, R)) \subset V(I)$. Therefore, $\text{E-Ass}(M) \subset \bigcup_{i=0}^2 \text{Ass}(\text{Ext}_R^i(M, R)) \cup V(I)$. The later is a finite set. So, $\text{E-Ass}(M)$ is a finite set.

ii) Reduced rings are $R(0)$. This implies that $\text{ht}(I) > 0$. The reminder of the proof go ahead as of part i). \square

Lemma 5.9. *Let R be a 2-dimensional local ring. Suppose $\text{E-Ass}(I)$ is finite for all unmixed ideals of codimension zero. Then $\text{E-Ass}(I)$ is finite for all I .*

Proof. We may assume that $\text{ht}(I) = 0$. Suppose I is not unmixed and the claim is hold for unmixed part. we write $I = I_1 \cap I_2$, where I_1 is the unmixed part and I_2 is the mixed part. As $\text{ht}(I_2) > 0$, then by the assumption $|\text{E-Ass}(R/I_2)| < \infty$. By the same reasoning, the claim is true for the ideal $I_1 + I_2$. We look

at the exact sequence $0 \rightarrow R/I \rightarrow R/I_1 \oplus R/I_2 \rightarrow R/(I_1 + I_2) \rightarrow 0$. This induces the following long exact sequence of Ext-modules

$$\dots \rightarrow \text{Ext}_R^{i-1}(R/(I_1 + I_2), R) \xrightarrow{\rho_i} \text{Ext}_R^i(R/I, R) \xrightarrow{\sigma_i} \text{Ext}_R^i(R/I_1, R) \oplus \text{Ext}_R^i(R/I_2, R) \rightarrow \dots$$

Although “ $|\text{Ass}| < \infty$ ” is not well-behaved with respect to the quotient, “ $|\text{Supp}| < \infty$ ” behaves well with respect to the quotient. Set $E_i := \frac{\text{Ext}_R^{i-1}(R/(I_1 + I_2), R)}{\ker(\rho_i)}$. In particular, $\bigcup_i \text{Supp}(E_i)$ is finite. Set $D_i := \text{im}(\sigma_i)$. Then $\bigcup_i \text{Ass}(D_i)$ is finite. Now look at the exact sequence $0 \rightarrow E_i \rightarrow \text{Ext}_R^i(R/I, R) \rightarrow D_i \rightarrow 0$. This implies that $\text{Ass}(\text{Ext}_R^i(R/I, R)) \subseteq \text{Ass}(E_i) \cup \text{Ass}(D_i)$, as claimed. Hence, without loss of generality we assume that I is unmixed and is of zero height. \square

Example 5.10. Look at the Fermat’s curve $\mathbb{F}_p[[X, Y, Z]]/(X^p + Y^p + Z^p)$. Then $\text{E-Ass}(I)$ is finite for all I .

Proof. In view of the above lemma, we assume I is unmixed and of codimension zero. Note that $R := \mathbb{F}_p[[X, Y, Z]]/(X + Y + Z)^p$. Set $\xi := (x + y + z)^i$ and $\zeta := (x + y + z)^{p-i}$. Without loss of the generality we assume that $I = \zeta R$. Its free resolution is given by:

$$\dots \xrightarrow{\zeta} R \xrightarrow{\xi} R \xrightarrow{\zeta} R \xrightarrow{\xi} I.$$

We deduce from this that $|\text{E-Ass}(I)| < \infty$. \square

Example 5.11. $R := \mathbb{F}[[X, Y, Z]]/(X^2, XYZ)$. Then $\text{E-Ass}(I)$ is finite for all ideal I .

Proof. In view of the above lemma, we may assume that $I = (x)$. Set $a := yz$. Note that $R_a \simeq \mathbb{F}[[y, z, y^{-1}, z^{-1}]]$ which is regular. It turns out that $\text{Ext}_R^i(I, R)$ is annihilated by some uniform power of a for all $i \geq 2$, see Fact 5.6. Thus $\text{Supp}(\text{Ext}_R^i(I, R)) \subset V(I + yz)$ for all $i \geq 2$. Note that $V(I + yz)$ is finite. So, $\text{E-Ass}_R(I) \subseteq \bigcup_{i=0}^1 \text{Ass}(\text{Ext}_R^i(I, R)) \cup V(I + yz)$, which is a finite set. \square

Remark 5.12. Let R be a normal closed Cohen-Macaulay local domain of dimension four. Suppose $|\text{E-Ass}(R/\mathfrak{b})| < \infty$ for almost complete-intersection ideal \mathfrak{b} of height 1. The following are equivalent:

- i) $|\text{E-Ass}(R/\mathfrak{b})| < \infty$ for all ideal \mathfrak{b} of height exactly two.
- ii) $|\text{E-Ass}(M)| < \infty$ for all finitely generated M .
- iii) $|\text{E-Ass}(R/\mathfrak{b})| < \infty$ for all unmixed ideal \mathfrak{b} of height exactly two.

Proof. In view of Corollary 5.7, we may assume that $\dim R > 2$.

$i) \implies ii)$: Keep the above lemma in mind and let \mathfrak{b} be an ideal of max-height at most two. If $\text{ht}(\mathfrak{b}) = 2$ we are done by the assumption. Suppose $\text{ht}(\mathfrak{b}) = 1$. We claim that things reduce to the unmixed case. To this end, we write $\mathfrak{b} = \mathfrak{b}_1 \cap \mathfrak{b}_2$, where \mathfrak{b}_1 is the unmixed part and \mathfrak{b}_2 is the mixed part. We assume that the claim is true for \mathfrak{b}_1 . If $\text{ht}(\mathfrak{b}_2) = 2$, then by the assumption $|\text{E-Ass}(R/\mathfrak{b}_2)| < \infty$. Thus, $\text{ht}(\mathfrak{b}_2) > 2$. Consequently, $\dim R/\mathfrak{b}_2 \leq 1$. Here, we used our low-dimensional assumption. This yields that $\text{E-Ass}(R/\mathfrak{b}_2) \subset V(\mathfrak{b}_2)$ which is a finite set. By the same reasoning, the claim is true for the ideal $\mathfrak{b}_1 + \mathfrak{b}_2$, because $\text{ht}(\mathfrak{b}_1 + \mathfrak{b}_2) > 1$. We look at the exact sequence $0 \rightarrow R/\mathfrak{b} \rightarrow R/\mathfrak{b}_1 \oplus R/\mathfrak{b}_2 \rightarrow R/(\mathfrak{b}_1 + \mathfrak{b}_2) \rightarrow 0$. This induces the following long exact sequence of Ext-modules

$$\dots \rightarrow \text{Ext}_R^{i-1}(R/(\mathfrak{b}_1 + \mathfrak{b}_2), R) \xrightarrow{\rho_i} \text{Ext}_R^i(R/\mathfrak{b}, R) \xrightarrow{\sigma_i} \text{Ext}_R^i(R/\mathfrak{b}_1, R) \oplus \text{Ext}_R^i(R/\mathfrak{b}_2, R) \rightarrow \dots$$

The reasoning given by Lemma 5.9 allow us to assume that \mathfrak{b} is unmixed and is of height one. In view of Fact 5.6, we may assume that \mathfrak{b} is not principal. Let b such that it generates \mathfrak{b} at the minimal primes of \mathfrak{b} . Look at the irredundant decomposition of $(b) = \mathfrak{b} \cap \mathfrak{c}$. Let c be in \mathfrak{c} but not in minimal primes of \mathfrak{b} .

If $\text{ht}(b, c) = 2$, then (b, c) is complete-intersection and the claim follows by Fact 5.6. So, (b, c) is almost complete intersection.

Claim A. One has $\mathfrak{b} = (b : c)$. Indeed, let $x \in \mathfrak{b}$. Then $xc \in \mathfrak{b} \cap \mathfrak{c} = (b)$. For the other side inclusion, let $x \in (b : c)$. Then $xc \in (b) = \bigcap q_i \cap \bigcap \tilde{q}_j$ where q_i (resp. \tilde{q}_j) are primary components of \mathfrak{b} (resp. \mathfrak{c}). Since \mathfrak{b} is unmixed, $\{\text{rad}(q_i)\}_i = \min(\mathfrak{b})$. Due to the choice of c , $c \notin \text{rad}(q_i)$ for all i . Since $xc \in q_i$ and $c \notin \text{rad}(q_i)$, we deduce from the definition of primary ideals that $x \in q_i$ for all i , i.e., $x \in (b)$. So, $(b : c) \subset \mathfrak{b}$. This proves the claim.

Claim A) induces the following exact sequence

$$0 \longrightarrow R/\mathfrak{b} \xrightarrow{c} R/(b) \longrightarrow R/(b, c) \longrightarrow 0.$$

Keep in mind that $\text{p.dim}(R/\mathfrak{b}R) = 1$. The induced long exact sequence tells us $\text{Ext}_R^i(R/\mathfrak{b}, -) \simeq \text{Ext}_R^{i+1}(R/(c, b), -)$ for all $i > 1$. This completes the proof of the implication $i) \implies ii)$.

$ii) \implies iii)$: This is trivial.

$iii) \implies i)$: Suppose $\text{ht}(\mathfrak{b}) = 2$. We need to reduce things to the unmixed part. The proof is a repetition of the above argument. We left it to the reader. \square

We left to the reader to find more reductions to almost complete-intersection ideals.

6. EXT AND THE NUMBER OF GENERATORS

Denote the minimal number of generators of a module by $\mu(-)$.

Theorem 6.1. (Macaulay 1904, Vasconcelos 1967, and Smith 2013) Let F be a field and I be an ideal in $S := F[X, Y]$ such that S/I is a Poincaré duality algebra. Then $\mu(I) = 2$.

Also, see [24, Proposition 2.4]. The Poincaré duality algebra is Gorenstein.

Theorem 6.2. (Serre 1960) Let S be a regular local ring and I be a height two ideal such that $R := S/I$ is Gorenstein. Then $\mu(I) = 2$.

The Gorenstein condition implies that $\omega_R = \text{Ext}_S^2(R, S)$ is cyclic.

Lemma 6.3. Let (R, \mathfrak{m}) be local and M be such that $p := \text{p.dim}(M) < \infty$. Then $\mu(\text{Ext}_R^p(M, R)) = \beta_p(M)$.

Proof. Recall that $\beta_p(-)$ is the Betti number. Look at the minimal free resolution of M

$$0 \longrightarrow R^{\beta_p(M)} \xrightarrow{X} R^{\beta_{p-1}(M)} \longrightarrow \dots \longrightarrow R^{\mu(M)} \longrightarrow M \longrightarrow 0.$$

Delete M from the right and apply $\text{Hom}_R(-, R)$ we arise to the following complex

$$0 \longrightarrow R^{\mu(M)} \longrightarrow \dots \longrightarrow R^{\beta_{p-1}(M)} \xrightarrow{X^t} R^{\beta_p(M)} \longrightarrow 0.$$

Here, $(-)^t$ denotes the transpose. Set $\mathbf{x}_i := (x_{i1}, \dots, x_{i\beta_p(M)})$. Thus $\text{Ext}_R^p(M, R) = \frac{R^{\beta_p(M)}}{(\mathbf{x}_1, \dots, \mathbf{x}_{\mu(M)})}$. This shows that $\mu(\text{Ext}_R^p(M, R)) \leq \mu(R^{\beta_p(M)})$. One has $(\mathbf{x}_1, \dots, \mathbf{x}_{\mu(M)})R^{\beta_p(M)} \subset \mathfrak{m}R^{\beta_p(M)}$. Consequently, $\mathfrak{m} \frac{R^{\beta_p(M)}}{(\mathbf{x}_1, \dots, \mathbf{x}_{\mu(M)})} = \frac{\mathfrak{m}R^{\beta_p(M)}}{(\mathbf{x}_1, \dots, \mathbf{x}_{\mu(M)})}$. By [16, Page 35], $\mu(E) = \dim_{R/\mathfrak{m}}(E/\mathfrak{m}E)$ for an R -module E . Therefore,

$$\begin{aligned} \mu(\text{Ext}_R^p(M, R)) &= \dim_{R/\mathfrak{m}} \left(\frac{R^{\beta_p(M)}}{(\mathbf{x}_1, \dots, \mathbf{x}_{\mu(M)})} / \frac{\mathfrak{m}R^{\beta_p(M)}}{(\mathbf{x}_1, \dots, \mathbf{x}_{\mu(M)})} \right) \\ &= \dim_{R/\mathfrak{m}} \left(\frac{R^{\beta_p(M)}}{\mathfrak{m}R^{\beta_p(M)}} \right) \\ &= \mu(R^{\beta_p(M)}) \\ &= \beta_p(M). \end{aligned}$$

□

Observation 6.4. Let (S, \mathfrak{m}) be a Cohen-Macaulay local ring and I a Cohen-Macaulay ideal of height two and of finite projective dimension. Then $\mu(\text{Ext}_S^2(S/I, S)) = \mu(I) - 1$.

Proof. Set $R := S/I$. By Auslander-Buchsbaum formula,

$$\text{p. dim}_S(R) = \text{depth } S - \text{depth}_S R = \text{depth } S - \text{depth}_R R = \dim S - \dim S/I = \text{ht}(I) = 2.$$

Set $\mu := \mu(I)$. By Hilbert-Burch, there is a matrix X with entries from \mathfrak{m} such that the minimal free resolution of S/I is

$$0 \longrightarrow S^{\mu-1} \xrightarrow{X} S^\mu \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

Thus $\beta_2(S/I) = \mu - 1$. In view of Lemma 6.3 we get the claim. □

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